Recitation 10. November 19

Focus: Singular Value Decomposition.

Recall that for a matrix A the **Singular Value Decomposition** (SVD) is an expression $A = U\Sigma V^T$ where U, V are orthogonal matrices and Σ is diagonal.

The **Singular Values** denoted σ_i are the diagonal entries of Σ.

The **Pseudo-inverse** of A is given in terms of the SVD by $A^+ = V\Sigma^+U^T$ where Σ^+ has diagonal entries $\frac{1}{\sigma_i}$.

A^+A and AA^+ are the projections onto $C(A^T)$ and $C(A)$ respectively.

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- Compute the Singular Value Decomposition of A.
- Compute the Psuedo-inverse A^+. Then compute the inverse A^{-1} by another method. How do they compare?

Solution: First we calculate $A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Next we diagonalize.

$$\det(A - \lambda Id) = \lambda^2 - 10\lambda + 16 \qquad \Rightarrow \qquad \lambda = 8, 2$$

And A^TA has orthonormal eigenvectors

$$N\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \mathbb{R}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad N\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \mathbb{R}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For $\lambda = 8, 2$ respectively. We therefore set

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Then to find the u_i we calculate

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 4 \\ 0 \end{bmatrix}\frac{1}{2\sqrt{2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $U = Id$ and the full SVD is

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then the Pseudo-inverse is

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}.$$

We can also use the formula for the inverse of a 2x2 matrix to see $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$. Of course, this agrees with the Pseudo-inverse because A is invertible.

2. 1. Find the maximum of the function

$$\frac{3x_1^2 + 2x_1x_2 + 3x_2^2}{x_1^2 + x_2^2}$$

   by expressing it in the form $\frac{x^TSx}{x^Tx}$ for a symmetric matrix S and using the relation of this expression to the eigenvalues of S. For what values of (x_1, x_2) is the maximum achieved?

   2. Find the minimum of the function

$$\sqrt{\frac{(x_1 + 4x_2)^2}{x_1^2 + x_2^2}}$$

by expressing it in the form $\frac{\|Ax\|}{\|x\|}$ and using the relation of this expression to the singular values of A.

Solution: We can express the numerator $3x_1^2+2x_1x_2+3x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so the expression is $\frac{x^TSx}{x^Tx}$ for $S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The maximum of the expression is given by the largest eigenvalue of the matrix, and the maximum is achieved at the corresponding eigenvector. The matrix has eigenvalues $\lambda = 4, 2$ so the maximum is 4. The corresponding eigenvector is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So the minimum is achieved at any multiple of this vector $\mathbb{R} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We can express $(x_1 + 4x_2)^2 = \|Ax\|^2$ for the 1×2 matrix $A = \begin{bmatrix} 1 & 4 \end{bmatrix}$. The expression is minimized by the smallest singular value of A (in absolute value), which is the square root of the smallest eigenvalue of

$$A^TA = \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$$

which is 0, since $A^TA \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 0$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

1. Compute its singular value decomposition
2. Use this to find the closest vector to $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the column space of A and in the column space of A^T. How else could you compute these vectors? Do the other methods agree?

Solution: The singular value decomposition is

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The pseudo-inverse is then

$$A^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The closest vector vector to the column space of A is then given by the projection $AA^+ = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ applied to the vector. Therefore the closest vector in the column space of A to $b = \begin{bmatrix} 3 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We could alternatively compute it by observing the column space of A is spanned by $\begin{bmatrix} 1 & 1 \end{bmatrix}$ and computing $P_{C(A)} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{b \cdot (1,1)}{|(1,1)|^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Of course these are the same vector. The second part is the same since $A = A^T$.

4. 1. If $A = QR$ is a Gram-Schmidt Orthogonalization of A (i.e. Q is an orthogonal matrix), how does the SVD of A relate to the SVD of R?
   2. If $A = U\Sigma V^T$ is a SVD of a matrix A, and Q_1, Q_2 are two orthogonal matrices, how do the singular values σ_i of $Q_1AQ_2^{-1}$ relate to those of A?

Solution:

1. Suppose $R = U\Sigma V^T$ is the SVD of R. Then $A = Q(U\Sigma V^T) = (QU)\Sigma V^T$. Since U and Q are both orthogonal matrices, so is QU, hence the latter is the SVD of A. That is to say, to obtain the SVD of A from that of R, we replace u_i with Qu_i and keep v_i and σ_i unchanged.

2. Suppose $A = U\Sigma V^T$ is the SVD of A. Then

$$Q_1 A Q_2^{-1} = Q_1(U\Sigma V^T)Q_2^T = (Q_1 U)\Sigma(Q_2 V)^T.$$

   In passing from the first to the second expressions we have used that $Q_2^{-1} = Q_2^T$ since Q_2 is orthogonal. As in the previous part, $Q_1 U$ and $Q_2 V$ are orthogonal since Q_1, Q_2, U, V are. Therefore the final expression is the SVD of $Q_1 A Q_2^{-1}$. In particular, we see the singular values σ_i are unchanged.